

# The $k$ -proper index of complete bipartite and complete multipartite graphs\*

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## Abstract

Let  $G$  be an edge-colored graph. A tree  $T$  in  $G$  is a *proper tree* if no two adjacent edges of it are assigned the same color. Let  $k$  be a fixed integer with  $2 \leq k \leq n$ . For a vertex subset  $S \subseteq V(G)$  with  $|S| \geq 2$ , a tree is called an  *$S$ -tree* if it connects  $S$  in  $G$ . A  *$k$ -proper coloring* of  $G$  is an edge-coloring of  $G$  having the property that for every set  $S$  of  $k$  vertices of  $G$ , there exists a proper  $S$ -tree  $T$  in  $G$ . The minimum number of colors that are required in a  $k$ -proper coloring of  $G$  is defined as the  *$k$ -proper index* of  $G$ , denoted by  $px_k(G)$ . In this paper, we determine the 3-proper index of all complete bipartite and complete multipartite graphs and partially determine the  $k$ -proper index of them for  $k \geq 4$ .

**Keywords:** 3-proper index, color code, binary system, complete bipartite and multipartite graphs,  $k$ -proper index.

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## 1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here. Let  $G$  be a graph, we use  $V(G)$ ,  $E(G)$ ,  $|G|$ ,  $\Delta(G)$  and  $\delta(G)$  to denote the vertex set, edge set, order (number of vertices), maximum degree and minimum degree of  $G$ , respectively. For  $D \subseteq V(G)$ , let  $\overline{D} = V(G) \setminus D$ , and let  $G[D]$  denote the subgraph of  $G$  induced by  $D$ .

Let  $G$  be a nontrivial connected graph with an *edge-coloring*  $c : E(G) \rightarrow \{1, \dots, t\}$ ,  $t \in \mathbb{N}$ , where adjacent edges may be colored with the same color. If adjacent edges of  $G$  receive different colors by  $c$ , then  $c$  is called a *proper coloring*. The minimum number of colors required in a proper coloring of  $G$  is referred as the *chromatic index* of  $G$  and

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denoted by  $\chi'(G)$ . Meanwhile, a path in  $G$  is called a *rainbow path* if no two edges of the path are colored with the same color. The graph  $G$  is called *rainbow connected* if for any two distinct vertices of  $G$ , there is a rainbow path connecting them. For a connected graph  $G$ , the *rainbow connection number* of  $G$ , denoted by  $rc(G)$ , is defined as the minimum number of colors that are required to make  $G$  rainbow connected. These concepts were first introduced by Chartrand et al. in [6] and have been well-studied since then. For further details, we refer the reader to a book [10].

Motivated by rainbow coloring and proper coloring in graphs, Andrews et al. [1] and, independently, Borożan et al. [3] introduced the concept of proper-path coloring. Let  $G$  be a nontrivial connected graph with an edge-coloring. A path in  $G$  is called a *proper path* if no two adjacent edges of the path are colored with the same color. The graph  $G$  is called *proper connected* if for any two distinct vertices of  $G$ , there is a proper path connecting them. The *proper connection number* of  $G$ , denoted by  $pc(G)$ , is defined as the minimum number of colors that are required to make  $G$  proper connected. For more details, we refer to a dynamic survey [9].

Chen et al. [7] recently generalized the concept of proper-path to proper tree. A tree  $T$  in an edge-colored graph is a *proper tree* if no two adjacent edges of it are assigned the same color. For a vertex subset  $S \subseteq V(G)$ , a tree is called an  *$S$ -tree* if it connects  $S$  in  $G$ . Let  $G$  be a connected graph of order  $n$  with an edge-coloring and let  $k$  be a fixed integer with  $2 \leq k \leq n$ . A  *$k$ -proper coloring* of  $G$  is an edge-coloring of  $G$  having the property that for every set  $S$  of  $k$  vertices of  $G$ , there exists a proper  $S$ -tree  $T$  in  $G$ . The minimum number of colors that are required in a  $k$ -proper coloring of  $G$  is the  *$k$ -proper index* of  $G$ , denoted by  $px_k(G)$ . Clearly,  $px_2(G)$  is precisely the proper connection number  $pc(G)$  of  $G$ . For a connected graph  $G$ , it is easy to see that  $px_2(G) \leq px_3(G) \leq \dots \leq px_n(G)$ . The following results are not difficult to get.

**Proposition 1.** [7] *If  $G$  is a nontrivial connected graph of order  $n \geq 3$ , and  $H$  is a connected spanning subgraph of  $G$ , then  $px_k(G) \leq px_k(H)$  for any  $k$  with  $3 \leq k \leq n$ . In particular,  $px_k(G) \leq px_k(T)$  for every spanning tree  $T$  of  $G$ .*

**Proposition 2.** [7] *For an arbitrary connected graph  $G$  with order  $n \geq 3$ , we have  $px_k(G) \geq 2$  for any integer  $k$  with  $3 \leq k \leq n$ .*

A *Hamiltonian path* in a graph  $G$  is a path containing every vertex of  $G$  and a graph having a Hamiltonian path is a *traceable graph*.

**Proposition 3.** [7] *If  $G$  is a traceable graph with  $n \geq 3$  vertices, then  $px_k(G) = 2$  for each integer  $k$  with  $3 \leq k \leq n$ .*

Armed with Proposition 3, we can easily get  $px_k(K_n) = px_k(P_n) = px_k(C_n) = px_k(W_n) = px_k(K_{s,s}) = 2$  for each integer  $k$  with  $3 \leq k \leq n$ , where  $K_n$ ,  $P_n$ ,  $C_n$  and  $W_n$  are respectively a complete graph, a path, a cycle and a wheel on  $n \geq 3$  vertices and  $K_{s,s}$  is a regular complete bipartite graph with  $s \geq 2$ .

A vertex set  $D \subseteq G$  is called an  *$s$ -dominating set* of  $G$  if every vertex in  $\overline{D}$  is adjacent to at least  $s$  distinct vertices of  $D$ . If, in addition,  $G[D]$  is connected, then we

call  $D$  a *connected  $s$ -dominating set*. Recently, Chang et al. [4] gave an upper bound for the 3-proper index of graphs with respect to the connected 3-dominating set.

**Theorem 1.** [4] *If  $D$  is a connected 3-dominating set of a connected graph  $G$  with minimum degree  $\delta(G) \geq 3$ , then  $px_3(G) \leq px_3(G[D]) + 1$ .*

Using this, we can easily get the following.

**Theorem 2.** *For any complete bipartite graph  $K_{s,t}$  with  $t \geq s \geq 3$ , we have  $2 \leq px_3(K_{s,t}) \leq 3$ .*

*Proof.* Let  $U$  and  $W$  be the two partite sets of  $K_{s,t}$ , where  $U = \{u_1, u_2, u_3, \dots, u_s\}$  and  $W = \{w_1, w_2, w_3, \dots, w_t\}$ . Obviously,  $D = \{u_1, u_2, u_3, w_1, w_2, w_3\}$  is a connected 3-dominating set of  $K_{s,t}$  and  $\delta(K_{s,t}) \geq 3$ . It follows from Theorem 1 that  $px_3(K_{s,t}) \leq px_3(G[D]) + 1 = 3$ . By Proposition 2, we have  $px_3(K_{s,t}) \geq 2$ .  $\square$

Naturally, we wonder among them, whose 3-proper index is 2. Moreover, what are the exact values of  $px_3(K_{s,t})$  with  $s + t \geq 3, t \geq s \geq 1$  and  $px_3(K_{n_1, n_2, \dots, n_r})$  with  $r \geq 3$ ? Moreover, what happens when  $k \geq 4$ ? So our paper is organised as follows: In Section 2, we concentrate on all complete bipartite graphs and determine the value of the 3-proper index of each of them. In Section 3, we go on investigating all complete multipartite graphs and obtain the 3-proper index of each of them. In the final section, we turn to the case that  $k \geq 4$ , and give a partial answer. In the sequel, we use  $c(uw)$  to denote the color of the edge  $uw$ .

## 2 The 3-proper index of a complete bipartite graph

In this section, we concentrate on all complete bipartite graphs  $K_{s,t}$  with  $s + t \geq 3, t \geq s \geq 1$  and get a complete answer of the value of  $px_3(K_{s,t})$ . From [7], we know  $px_3(K_{1,t}) = t$ . Hence, in the following we assume that  $t \geq s \geq 2$ . Our result will be divided into three separate theorems depending upon the value of  $s$ .

**Theorem 3.** *For any integer  $t \geq 2$ , we have*

$$px_3(K_{2,t}) = \begin{cases} 2 & \text{if } 2 \leq t \leq 4; \\ 3 & \text{if } 5 \leq t \leq 18; \\ \lceil \sqrt{\frac{t}{2}} \rceil & \text{if } t \geq 19. \end{cases}$$

**Proof.** Let  $U, W$  be the two partite sets of  $K_{2,t}$ , where  $U = \{u_1, u_2\}$  and  $W = \{w_1, w_2, \dots, w_t\}$ . Suppose that there exists a 3-proper coloring  $c : E(K_{2,t}) \rightarrow \{1, 2, \dots, k\}$ ,  $k \in \mathbb{N}$ . Corresponding to the 3-proper coloring, there is a color code  $(w)$  assigned to every vertex  $w \in W$ , consisting of an ordered 2-tuple  $(a_1, a_2)$ , where  $a_i = c(u_i w) \in \{1, 2, \dots, k\}$  for  $i = 1, 2$ . In turn, if we give each vertex of  $W$  a code, then we can induce the corresponding edge-coloring of  $G$ .

**Claim 1:**  $px_3(K_{2,t}) = 2$  if  $2 \leq t \leq 4$ .

*Proof.* Give the codes  $(1, 2), (2, 1), (1, 1), (2, 2)$  to  $w_1, w_2, w_3, w_4$  (if there is). Then it is easy to check that for every 3-subset  $S$  of  $K_{2,t}$ , the edge-colored  $K_{2,t}$  has a proper path  $P$  connecting  $S$ .  $\square$

**Claim 2:**  $px_3(K_{2,t}) > 2$  if  $t > 4$ .

*Proof.* Otherwise, give  $K_{2,t}$  a 3-proper coloring with colors 1 and 2. Then for any 3-subset  $S$  of  $K_{2,t}$ , any proper tree connecting  $S$  must be a path, actually. For  $t > 4$ , there are at least two vertices  $w_p, w_q$  in  $W$  such that  $code(w_p) = code(w_q)$ . We may assume that  $code(w_1) = code(w_2)$ . Then for an arbitrary integer  $i$  with  $3 \leq i \leq t$ , let  $S = \{w_1, w_2, w_i\}$ . There must be a proper path of length 4 connecting  $S$ . Suppose that the path is  $w_a u_{a'} w_b u_{b'} w_c$ , where  $\{w_a, w_b, w_c\} = \{w_1, w_2, w_i\}$  and  $\{u_{a'}, u_{b'}\} = \{u_1, u_2\}$ . By symmetry, we can assume that  $u_{a'} = u_1, u_{b'} = u_2$ . Then  $w_b = w_i$  for otherwise we have  $c(w_a u_1) = c(u_1 w_b)$  or  $c(w_b u_2) = c(u_2 w_c)$ , a contradiction. For equivalence, let  $w_a = w_1, w_c = w_2$ . Thus  $c(w_i u_1) \neq c(w_i u_2)$ . Without loss of generality, we can suppose that  $c(w_i u_1) = 1$  and  $c(w_i u_2) = 2$ . Hence,  $code(w_i) = (1, 2)$  for each integer  $3 \leq i \leq t$ . Now let  $S = \{w_3, w_4, w_5\}$ . It is easy to verify that there is no proper path  $w_a u_{a'} w_b u_{b'} w_c$  connecting  $S$ , for we always have  $c(w_a u_{a'}) = c(u_{a'} w_b), c(w_b u_{b'}) = c(u_{b'} w_c)$ .  $\square$

**Claim 3:** Let  $k$  be a integer where  $k \geq 3$ . Then  $px_3(K_{2,t}) \leq k$  for  $4 < t \leq 2k^2$ .

*Proof.* Set  $code(w_1) = (1, 1), code(w_2) = (1, 2), \dots, code(w_k) = (1, k);$

$code(w_{k+1}) = (2, 1), code(w_{k+2}) = (2, 2), \dots, code(w_{2k}) = (2, k);$

$\dots$

$code(w_{k(k-1)+1}) = (k, 1), code(w_{k(k-1)+2}) = (k, 2), \dots, code(w_{k^2}) = (k, k)$

(if there is). And let  $code(w_{k^2+i}) = code(w_i)$  for  $1 \leq i \leq k^2$  (if there is). Now, we prove that this is a 3-proper coloring of  $K_{2,t}$ . First of all, we notice that each code appears at most twice. Let  $S$  be a 3-subset of  $K_{2,t}$ . We consider the following two cases.

**Case 1:** Let  $S = \{w_l, w_m, w_n\}$ , where  $1 \leq l < m < n \leq t$ .

**Subcase 1.1:** If there is a  $j \in \{1, 2\}$  such that the colors of  $u_j w_l, u_j w_m, u_j w_n$  are pairwise distinct, then the tree  $T = \{u_j w_l, u_j w_m, u_j w_n\}$  is a proper  $S$ -tree.

**Subcase 1.2:** If there is no such  $j$ , that is, at least two of the edges  $u_j w_l, u_j w_m, u_j w_n$  share the same color for both  $j = 1$  and  $j = 2$ .

*i)*  $code(w_l), code(w_m)$  and  $code(w_n)$  are pairwise distinct. Without loss of generality, we suppose that  $c(u_1 w_l) = c(u_1 w_m) = a, c(u_2 w_l) = c(u_2 w_n) = b$  ( $1 \leq a, b \leq k^2$ ). Then  $c(u_1 w_n) \neq c(u_1 w_l), c(u_2 w_l) \neq c(u_2 w_m)$ . If  $a = b$ , then we have  $c(u_1 w_n) \neq c(w_n u_2)$ . So the path  $P = w_l u_1 w_n u_2 w_m$  is a proper  $S$ -tree. Otherwise, the path  $P = w_n u_1 w_l u_2 w_m$  is a proper  $S$ -tree.

*ii)* Two of the codes of the vertices in  $S$  are the same. Without loss of generality, we assume that  $code(w_l) = code(w_m) = (a, b), code(w_n) = (x, y)$  ( $1 \leq a, b, x, y \leq k^2$ ). Notice that  $(x, y) \neq (a, b)$ , then suppose that  $x \neq a$ . Since  $k \geq 3$ , there are two positive integers  $p, q \leq k$  such that  $p \neq a, p \neq x$  and  $q \neq b, q \neq y$ . Pick a vertex  $w_r$  whose code

is  $(p, q)$  (this vertex exists since all of the  $k^2$  codes appear at least once). Then the tree  $T = \{u_1w_m, u_1w_n, u_1w_r, w_ru_2, u_2w_l\}$  is a proper  $S$ -tree.

**Case 2:**  $S = \{u_r, w_l, w_m\}$ , where  $1 \leq l < m \leq t$ . By symmetry, let  $r = 1$ .

Suppose that  $\text{code}(w_l) = (a, b), \text{code}(w_m) = (x, y)$  ( $1 \leq a, b, x, y \leq k^2$ ). If  $a \neq x$  then the path  $P = w_lu_1w_m$  is a proper  $S$ -tree. If  $a = x$ , then we consider whether  $b = y$  or not. We discuss two subcases.

i)  $b \neq y$ , then at least one of them is not equal to  $a$ , assume that  $b \neq a$ . So the path  $P = u_1w_lu_2w_m$  is a proper  $S$ -tree.

ii)  $b = y$ , that is  $\text{code}(w_l) = \text{code}(w_m)$ , so all of the  $k^2$  codes appear at least at once. Since  $k \geq 3$ , there are two positive integers  $p, q \leq k$  such that  $p \neq a$  and  $q \neq b, q \neq p$ . Pick a vertex  $w_r$  whose code is  $(p, q)$ . Then the path  $P = w_lu_1w_ru_2w_m$  is a proper  $S$ -tree.

**Case 3:**  $S = \{u_1, u_2, w_l\}$ , where  $1 \leq l \leq t$ .

Suppose that  $\text{code}(w_l) = (a, b)$  ( $1 \leq a, b \leq k^2$ ). If  $a \neq b$ , then the path  $P = u_1w_lu_2$  is a proper  $S$ -tree. Otherwise, according to our edge-coloring, there exists a vertex  $w_r$  of  $W$  with the code  $(p, q)$  such that  $q \neq a$  and  $p \neq q$ . Then the path  $P = w_lu_2w_ru_1$  is a proper  $S$ -tree.  $\square$

**Claim 4:**  $px_3(K_{2,t}) > k$  for  $t > 2k^2$ .

*Proof.* For any edge-coloring of  $K_{2,t}$  with  $k$  colors, there must be a code which appears at least three times. Suppose that  $w_1, w_2, w_3$  are the vertices with the same code and set  $S = \{w_1, w_2, w_3\}$ . Then for any tree  $T$  connecting  $S$ , there is a  $j \in \{1, 2\}$  such that  $\{u_jw_l, u_jw_m\} \subseteq E(T)$  for some  $\{l, m\} \subseteq \{1, 2, 3\}, l \neq m$ . But  $c(u_jw_l) = c(u_jw_m)$ , so  $T$  can not be a proper  $S$ -tree. Thus  $px_3(K_{2,t}) > k$ .  $\square$

By Claims 2-4, we have the following result: if  $5 \leq t \leq 8$ ,  $px_3(K_{2,t}) = 3$ ; if  $t > 8$ , let  $k = \lceil \sqrt{\frac{t}{2}} \rceil$ , then  $3 \leq \sqrt{\frac{t}{2}} \leq k < \sqrt{\frac{t}{2}} + 1$ , i.e.,  $2(k-1)^2 + 1 \leq t \leq 2k^2$ , so we have  $px_3(K_{2,t}) = k = \lceil \sqrt{\frac{t}{2}} \rceil$ . Notice that  $px_3(K_{2,t}) = 3$  for  $5 \leq t \leq 18$ .  $\blacksquare$

**Theorem 4.** For any integer  $t \geq 3$ , we have

$$px_3(K_{3,t}) = \begin{cases} 2 & \text{if } 3 \leq t \leq 12; \\ 3 & \text{otherwise.} \end{cases}$$

**Proof.** Let  $U, W$  be the two partite sets of  $K_{3,t}$ , where  $U = \{u_1, u_2, u_3\}$  and  $W = \{w_1, w_2, \dots, w_t\}$ . Suppose that there exists a 3-proper coloring  $c : E(K_{2,t}) \rightarrow \{0, 1, 2, \dots, k-1\}$ ,  $k \in \mathbb{N}$ . Analogously to Theorem 3, corresponding to the 3-proper coloring, there is a color  $\text{code}(w)$  assigned to every vertex  $w \in W$ , consisting of an ordered 3-tuple  $(a_1, a_2, a_3)$ , where  $a_i = c(u_iw) \in \{0, 1, 2, \dots, k-1\}$  for  $i = 1, 2, 3$ . In turn, if we give each vertex of  $W$  a code, then we can induce the corresponding edge-coloring of  $G$ .

**Case 1:**  $3 \leq t \leq 8$ .

In this part, we give the vertices of  $W$  the codes which induce a 3-proper coloring of  $K_{3,t}$  with colors 0 and 1. And by application of binary system, we can introduce the assignment of the codes in a clear way. Recall the Abelian group  $\mathbb{Z}_2$ . We build a bijection  $f : \{w_1, w_2, \dots, w_8\} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ , where  $f(w_{4a_1+2a_2+a_3+1}) = (a_1, a_2, a_3)$ . For instance,  $f(w_3) = (0, 1, 0)$ . Under this condition, we use its restriction  $f_W$  on  $W$ . Now, we prove that  $f$  induces a 3-proper coloring of  $K_{3,t}$ . Let  $S$  be an arbitrary 3-subset.

**Subcase 1.1:**  $S = \{w_l, w_m, w_n\}$ .

Because there is no copy of any code, we can find a vertex in  $U$ , say  $u_1$ , such that  $u_1w_l, u_1w_m, u_1w_n$  are not all with the same color. We may assume that  $c(u_1w_l) = c(u_1w_m) = 0$  and  $c(u_1w_n) = 1$ .

*i)*  $code(w_l) = (0, 0, 0)$ . Then there is a '1' in the code of  $w_m$ . By symmetry, assume that  $c(u_2w_m) = 1$ . Then there is a proper path  $P = w_lu_2w_mu_1w_n$  connecting  $S$ .

*ii)*  $code(w_l) = (0, 0, 1)$ . If  $code(w_m) = (0, 0, 0)$ , then we return to *i)*. Otherwise, the code of  $w_m$  is neither  $(0, 0, 0)$  nor  $(0, 0, 1)$ . So  $c(u_2w_m) = 1$ . Then the proper  $S$ -tree is the same as that in *i)*.

*iii)*  $code(w_l) = (0, 1, 0)$ . It is similar to *ii)*.

*iv)*  $code(w_l) = (0, 1, 1)$ . Then either  $c(u_2w_m) = 0$  or  $c(u_3w_m) = 0$ . By symmetry, we suppose that  $c(u_2w_m) = 0$ . Then the path  $P = w_mu_2w_lu_1w_n$  is a proper  $S$ -tree.

**Subcase 1.2:**  $S = \{u_j, w_l, w_m\}$ .

If  $c(u_jw_l) \neq c(u_jw_m)$ , then the path  $P = w_lu_jw_m$  is a proper  $S$ -tree. Otherwise, by symmetry, we assume that  $c(u_jw_l) = c(u_jw_m) = 0$ , then there is a  $j' \neq j$  such that  $c(u_{j'}w_l) \neq c(u_{j'}w_m)$  (otherwise  $w_l, w_m$  will have the same code). So one of  $c(u_{j'}w_l)$  and  $c(u_{j'}w_m)$  equals 1, say  $c(u_{j'}w_l) = 1$ . Then the path  $P = u_jw_lu_{j'}w_m$  is a proper  $S$ -tree.

**Subcase 1.3:**  $S = \{u_{j_1}, u_{j_2}, w_l\}$ .

If  $c(u_{j_1}w_l) \neq c(u_{j_2}w_l)$ , then the path  $P = u_{j_1}w_lu_{j_2}$  is a proper  $S$ -tree. Otherwise, by symmetry, we assume that  $c(u_{j_1}w_l) = c(u_{j_2}w_l) = 0$ . By the sequence of the codes according to  $f$  and  $t \geq 3$ , we know that for any two vertices  $u_{a'}, u_{b'}$  of  $U$ , there exists a vertex  $w \in W$  such that  $c(u_{a'}w) \neq c(u_{b'}w)$ . Similar to Subcase 1.2, we can obtain a proper  $S$ -tree.

**Subcase 1.4:**  $S = \{u_1, u_2, u_3\}$ .

$P = u_1w_3u_2w_2u_3$  is a proper path connecting  $S$ .

**Case 2:**  $9 \leq t \leq 12$ .

Set  $code(w_1) = (0, 0, 1), code(w_2) = (0, 1, 0), code(w_3) = (0, 1, 1),$

$code(w_4) = (1, 0, 0), code(w_5) = (1, 0, 1), code(w_6) = (1, 1, 0).$

And let  $code(w_{6+i}) = code(w_i)$  for  $1 \leq i \leq 6$  (if there is). For convenience, we denote  $w_{6+i} = w'_i$ . Now, we claim that this induces a 3-proper coloring of  $K_{3,t}$ . Let  $S$  be an arbitrary 3-subset of  $K_{3,t}$ . Based on Case 1, we only consider about the case that  $\{w_i, w'_i\} \subseteq S$  for some  $1 \leq i \leq 6$ . By symmetry, we suppose that  $i = 1$ . First of all, we

list three proper paths containing  $w_1, w'_1$ :  $P_1 = w_1u_3w_2u_2w'_1$ ,  $P_2 = w_1u_2w_3u_1w_4u_3w'_1$  and  $P_3 = w_1u_1w_5u_2w_6u_3w'_1$ , in which  $w_j$  can be replaced by  $w'_j$  for  $2 \leq j \leq 6$ . Then, we can always find a proper path from  $\{P_1, P_2, P_3\}$  connecting  $S$  whichever the third vertex of  $S$  is.

**Case 3:**  $t \geq 13$ .

Take into account Theorem 2, we claim that  $px_3(K_{3,t}) = 3$ . We prove it by contradiction. If there is a 3-proper coloring of  $K_{3,t}$  with two colors 0 and 1, then any proper tree for an arbitrary 3-subset  $S$  is in fact a path. Consider about the set  $S \subseteq W$ . As the graph is bipartite and we just care about the shortest proper path connecting  $S$ , there are only two possible types of such a path:

I:  $w_a u_{a'} w_b u_{b'} w_c$

II:  $w_a u_{a'} w_b u_{b'} w' u_{c'} w_c$

where  $\{u_{a'}, u_{b'}, u_{c'}\} = U$  and  $\{w_a, w_b, w_c\} = S, w' \in W \setminus S$ .

Firstly, as  $t \geq 13$ , we know that some code appears more than once. But it can not appear more than twice. Otherwise, suppose that  $w_i, w'_i, w''_i$  are the three vertices with the same code, and let  $S = \{w_i, w'_i, w''_i\}$ . Whether the proper path connecting  $S$  is type I or type II, it should be  $c(w_a u_{a'}) \neq c(w_b u_{a'})$ , contradicting with the same code of the three vertices.

Secondly, we prove the following several claims by contradiction.

**Claim 1:** The repetitive code can not be  $(0, 0, 0)$  or  $(1, 1, 1)$ .

*Proof.* Suppose that  $code(w_1) = code(w_2) = (0, 0, 0)$ . Let  $S = \{w_1, w_2, w_3\}$  where  $w_3 \in W \setminus \{w_1, w_2\}$ , and let  $P$  be a proper path connecting  $S$ . Then  $w_1, w_2$  are the two end vertices of  $P$ , and so the two end edges of it are assigned the same color. However, since the length of  $P$  is even, the colors of the end edges can not be the same, a contradiction. Analogously, the code  $(1, 1, 1)$  cannot appear more than once.  $\square$

**Claim 2:** If the code  $(0, 0, 1)$  is repeated, then there is no vertex in  $W$  with  $(0, 0, 0)$  as its code.

*Proof.* Suppose that  $code(w_1) = code(w_2) = (0, 0, 1), code(w_3) = (0, 0, 0)$ . Let  $S = \{w_1, w_2, w_3\}$ , and let  $P$  be a proper path connecting  $S$ . Then  $w_3$  is one of the end vertices of  $P$ . Moreover, the path  $P$  must be type II, for in type I, we need  $c(w_a u_{a'}) \neq c(w_b u_{a'})$  and  $c(w_b u_{b'}) \neq c(w_c u_{b'})$ , which is impossible for  $S$ . We can also deduce that  $u_{a'} = u_3$  because  $c(w_a u_{a'}) \neq c(w_b u_{a'})$ . And  $\{w_1, w_2\} \neq \{w_a, w_b\}$  since they are with the same code. So we have  $w_a = w_3$ . Thus,  $\{w_b, w_c\} = \{w_1, w_2\}$  and  $\{u_{b'}, u_{c'}\} = \{u_1, u_2\}$ , contradicting with the fact that  $c(w_b u_{b'}) \neq c(w_c u_{c'})$ .  $\square$

Analogously, we have that the repetitive code  $(0, 1, 0)$  or  $(1, 0, 0)$  can not exist along with the code  $(0, 0, 0)$ , respectively. And the repetitive code  $(0, 1, 1), (1, 0, 1)$  or  $(1, 1, 0)$  can not exist along with the code  $(1, 1, 1)$ , respectively.

Finally, as  $t \geq 13$  and no code could appear more than twice, there are at least 7 different codes in  $W$  and at least 5 codes repeated. But considering about Claim 2 and its analogous results, it is a contradiction. So  $px_3(K_{3,t}) = 3$  when  $t \geq 13$ . ■

**Theorem 5.** *For a complete bipartite graph  $K_{s,t}$  with  $t \geq s \geq 4$ , we have  $px_3(K_{s,t}) = 2$ .*

**Proof.** Let  $U, W$  be the two partite sets of  $K_{s,t}$ , where  $U = \{u_1, u_2, \dots, u_s\}$  and  $W = \{w_1, w_2, \dots, w_t\}$ . And denote a cycle  $C_s = u_1 w_1 u_2 w_2 \dots u_s w_s u_1$ . Moreover, if  $u, v \in V(C_s)$ , then we use  $uC_s v$  to denote the segment of  $C_s$  from  $u$  to  $v$  in the clockwise direction, otherwise we denote it by  $uC'_s v$ . Then we demonstrate a 3-proper coloring of  $K_{s,t}$  with two colors 0 and 1. Let  $c(u_i w_i) = 0$  ( $1 \leq i \leq s$ ) and  $c(u_i w_j) = 1$  ( $1 \leq i \neq j \leq s$ ). And assign  $c(w_r u_i) = i \pmod{2}$  ( $1 \leq i \leq s, s < r \leq t$ ). Now we prove that this coloring is a 3-proper coloring of  $K_{s,t}$ . Consider about its 3-subset  $S$ .

i)  $S \subseteq V(C_s)$ . The proper path is in  $C_s$ .

ii)  $S = \{w_l, w_m, w_n\}$  where  $l, m, n > s$ . Then the path  $P = w_l u_1 w_1 u_2 w_m u_3 w_3 u_4 w_n$  is a proper  $S$ -tree.

iii)  $S = \{w_l, w_m, w_n\}$  where  $l \leq s, m, n > s$ . If  $c(w_m u_l) = 1$ , then the path  $P = w_m u_l w_l C_s u_2 w_n$  is a proper  $S$ -tree. If  $c(w_m u_l) = 0$ , then the proper  $S$ -tree is the path  $P = w_m u_l w_{l-1} u_{l-1} w_n u_{l-2} C'_s w_l$ , where  $u_0 = u_s, u_{-1} = u_{s-1}$  if  $i_1 = 2$ .

iv)  $S = \{u_j, w_l, w_m\}$  where  $l, m > s$ . The way to find a proper  $S$ -tree is similar with that in iii).

v)  $S = \{u_j, w_l, w_m\}$  where  $l \leq s, m > s$ . If  $c(w_m u_j) = 1$ , then the proper  $S$ -tree is the path  $P = w_m u_j w_j C_s w_l$ . If  $c(w_m u_j) = 0$ , then the path  $P = w_m u_j C'_s w_l$  is a proper  $S$ -tree.

vi)  $S = \{u_{j_1}, u_{j_2}, w_i\}$  where  $i > s$ . The way to find a proper  $S$ -tree is similar with that in v). ■

**Remarks.** Here, we introduce a generalization of  $k$ -proper index which was proposed by Chang et. al. in [5] recently. Let  $G$  be a nontrivial  $\kappa$ -connected graph of order  $n$ , and let  $k$  and  $\ell$  be two integers with  $2 \leq k \leq n$  and  $1 \leq \ell \leq \kappa$ . For  $S \subseteq V(G)$ , let  $\{T_1, T_2, \dots, T_\ell\}$  be a set of  $S$ -tree, they are *internally disjoint* if  $E(T_i) \cap E(T_j) = \emptyset$  and  $V(T_i) \cap V(T_j) = S$  for every pair of distinct integers  $i, j$  with  $1 \leq i, j \leq \ell$ . The  $(k, \ell)$ -proper index of  $G$ , denoted by  $px_{k,\ell}(G)$ , is the minimum number of colors that are required in an edge-coloring of  $G$  such that for every  $k$ -subset  $S$  of  $V(G)$ , there exist  $\ell$  internally disjoint proper  $S$ -trees connecting them. In their paper, they investigated the complete bipartite graphs and obtained the following.

**Theorem 6.** [5] *Let  $s$  and  $t$  be two positive integers with  $t = O(s^r), r \in \mathbb{R}$  and  $r \geq 1$ . For every pair of integers  $k, \ell$  with  $k \geq 3$ , there exists a positive integer  $N_3 = N_3(k, \ell)$  such that  $px_{k,\ell}(K_{s,t}) = 2$  for every integer  $s \geq N_3$ .*

Obviously, they did not give the exact value of  $px_{k,\ell}(K_{s,t})$ , even for  $k = 3$  and  $\ell = 1$ . Our Theorem 5 completely determines the value of  $px_{k,\ell}(K_{s,t})$  for  $k = 3$  and  $\ell = 1$ , without using the condition that  $t = O(s^r), r \in \mathbb{R}$  and  $r \geq 1$ .



### 3 The 3-proper index of a complete multipartite graph

With the aids of Theorems 3, 4 and 5, we are now able to determine the 3-proper index of all complete multipartite graphs. First of all, we give a useful theorem.

**Theorem 7.** [8] *Let  $G$  be a graph with  $n$  vertices. If  $\delta(G) \geq \frac{n-1}{2}$ , then  $G$  has a Hamiltonian path.*

**Theorem 8.** *Let  $G = K_{n_1, n_2, \dots, n_r}$  be a complete multipartite graph, where  $r \geq 3$  and  $n_1 \leq n_2 \leq \dots \leq n_r$ . Set  $s = \sum_{i=1}^{r-1} n_i$  and  $t = n_r$ . Then we have*

$$px_3(G) = \begin{cases} 3 & \text{if } G = K_{1,1,t}, 5 \leq t \leq 18 \\ & \text{or } G = K_{1,2,t}, t \geq 13 \\ & \text{or } G = K_{1,1,1,t}, t \geq 15; \\ \lceil \sqrt{\frac{t}{2}} \rceil & \text{if } G = K_{1,1,t}, t \geq 19; \\ 2 & \text{otherwise.} \end{cases}$$

**Proof.** The graph  $G$  has a  $K_{s,t}$  as its spanning subgraph, so it follows from Propositions 1 and 2 that  $2 \leq px_3(G) \leq px_3(K_{s,t})$ . In the following, we discuss two cases according to the relationship between  $s$  and  $t$ .

**Case 1:**  $s \leq t$ . Let  $U_1, U_2, \dots, U_r$  denote the different  $r$ -partite sets of  $G$ , where  $|U_i| = n_i$  for each integer  $1 \leq i \leq r$ .

When  $s \geq 4$ , then by Theorem 5, we have  $px_3(G) = px_3(K_{s,t}) = 2$ . When  $s \leq 3$ , there are only three possible values of  $(n_1, n_2, \dots, n_{r-1})$ .

**Subcase 1:**  $(n_1, n_2, \dots, n_{r-1}) = (1, 1)$ . Set  $U_1 = \{u_1\}, U_2 = \{u_2\}$ . Under this condition, giving the edge  $u_1u_2$  an arbitrary color, the proof is exactly the same as that of Theorem 3. So it holds that  $px_3(G) = px_3(K_{2,t})$ .

**Subcase 2:**  $(n_1, n_2, \dots, n_{r-1}) = (1, 2)$ . Set  $U_1 = \{u_1\}, U_2 = \{u_2, u_3\}$  and  $W = U_r$ . By Theorem 4, we have  $px_3(G) = px_3(K_{3,t}) = 2$  if  $t \leq 12$ ;  $px_3(G) \leq px_3(K_{3,t}) = 3$  if  $t > 12$ . We claim that  $px_3(G) = 3$  if  $t > 12$ . Assume, to the contrary, that  $G$  has a 3-proper coloring with two colors 0 and 1. By symmetry, we assume that  $c(u_1u_2) = 0$ . With the similar reason in Case 3 of the proof of Theorem 4, no code can appear more than twice. And recall the bijection  $f$  defined in that proof. To label the vertices in  $W$ , we use its inverse  $f^{-1} : (a_1, a_2, a_3) \mapsto w_{4a_1+2a_2+a_3+1}$ , and denote by  $w'_i$  the copy of the vertex  $w_i$  with  $1 \leq i \leq 8$ . Then we prove the following results by contradiction.

**Claim 1:**  $\{w_1, w'_1, w_2\} \not\subseteq W$  and  $\{w_2, w'_2, w_1\} \not\subseteq W$ .

*Proof.* Set  $S = \{w_1, w'_1, w_2\}$ . We know from the proof of Theorem 4 that there is no proper path of type I or II. So the proper path  $P$  connecting  $S$  is type III:  $w_a u_{a'} w_b u_{b'} u_{c'} w_c$ . Then  $w_1, w'_1$  must be the end vertices of  $P$ , and so  $w_b = w_2$  and  $u_{a'} = u_3$ . Since  $c(w_a u_{a'}) = 0$ ,  $c(u_{b'} u_{c'}) = 1$ , contradicting with  $c(u_1 u_2) = 0$ . Hence, we get  $\{w_1, w'_1, w_2\} \not\subseteq W$ . Similarly, we have  $\{w_2, w'_2, w_1\} \not\subseteq W$ .  $\square$

**Claim 2:**  $\{w_4, w'_4, w_8\} \not\subseteq W$  and  $\{w_8, w'_8, w_4\} \not\subseteq W$ .

*Proof.* Set  $S = \{w_4, w'_4, w_8\}$ . Similar to Claim 1, any proper path  $P$  connecting  $S$  should be type III:  $w_a u_{a'} w_b u_{b'} u_{c'} w_c$ . Then  $w_8$  must be an end vertex of  $P$ , and so both of the end edges of  $P$  are colored with 1. Thus  $u_{a'} = u_1$ . Then  $\{u_{b'}, u_{c'}\} = \{u_2, u_3\}$  and  $c(u_2 u_3) = 0$ , contradicting with the fact that  $u_2 u_3 \notin E(G)$ . Similarly, we have  $\{w_8, w'_8, w_4\} \not\subseteq W$ .  $\square$

So there are four cases that some vertices can not exist in  $W$  at the same time, and each code appears at most twice. However, there are more than 12 vertices in  $W$ , a contradiction. So  $px_3(G) = px_3(K_{3,t}) = 3$  when  $t > 12$ .

**Subcase 3:**  $(n_1, n_2, \dots, n_{r-1}) = (1, 1, 1)$ . Set  $U = \cup_{j=1}^{r-1} U_j = \{u_1, u_2, u_3\}$  and  $W = U_r$ .

**Claim 3:**  $px_3(G) = 2$  if  $t \leq 14$ .

*Proof.* By Theorem 4, we have  $px_3(G) = px_3(K_{3,t}) = 2$  if  $t \leq 12$ ;  $px_3(G) \leq px_3(K_{3,t}) = 3$  if  $t > 12$ . When  $t = 13$  or  $14$ , we recall  $code(w)$  defined in Case 2 of Theorem 4. Set

$$\begin{aligned} code(w_1) &= (0, 0, 1), code(w_2) = (0, 1, 0), code(w_3) = (0, 1, 1), code(w_4) = (1, 0, 0), \\ code(w_5) &= (1, 0, 1), code(w_6) = (1, 1, 0), code(w_7) = (1, 1, 1). \end{aligned}$$

And let  $code(w_{7+i}) = code(w_i)$  for  $1 \leq i \leq 7$  (if there is) and  $c(u_i u_j) = 0$  for  $1 \leq i \neq j \leq 3$ . For convenience, we denote  $w_{7+i} = w'_i$ . Now, we claim that this induces a 3-proper coloring of  $G$ . Let  $S$  be an arbitrary 3-subset of  $G$ . Based on Theorem 4, we only consider about the case that  $w_7(w'_7) \in S$ . When  $S = \{w_1, w_7, w'_7\}$ , then the path  $P = w_7 u_1 w_1 u_3 u_2 w'_7$  is a proper path connecting  $S$ . Similarly, we can find a proper path in type III connecting  $S$  whichever the two other vertices of  $S$  are.  $\square$

**Claim 4:**  $px_3(G) = 3$  if  $t > 14$ .

*Proof.* Assume, to the contrary, that  $G$  has a 3-proper coloring with two colors 0 and 1. If the edges of  $G[U]$  are colored with two different colors, then we set  $u_2$  the common vertex of two edges with two different colors. Moreover, without loss of generality, we suppose that  $c(u_1 u_2) = 0$ . Similar to Subcase 2, we have  $px_3(G) = 3$  if  $t > 12$ . If all the edges of  $G[U]$  are colored with one color, say 0. Repeat the discussion in Subcase 2, then we know Claim 1 is also true under this condition. As  $t \geq 15$  and no code could appear more than twice, there are at least 8 different codes in  $W$  and at least 7 codes repeated. But from Claim 1, we know  $\{w_1, w'_1, w_2\} \not\subseteq W$  and  $\{w_2, w'_2, w_1\} \not\subseteq W$ . So  $px_3(G) = 3$  when  $t \geq 15$ .  $\square$

**Case 2:**  $s \geq t$ . Under this condition, we have  $\delta(G) \geq \frac{n-1}{2}$ . By Theorem 7, we know  $G$  is traceable. Thus, it follows from Proposition 3 that  $px_3(G) = 2$ .  $\blacksquare$

## 4 The $k$ -proper index

Now, we turn to the  $k$ -proper index of a complete bipartite graph and a complete multipartite graph for general  $k$ . Throughout this section, let  $k$  be a fixed integer with  $k \geq 3$ . Firstly, we generalize Theorem 1 to the  $k$ -proper index.

**Theorem 9.** *If  $D$  is a connected  $k$ -dominating set of a connected graph  $G$  with minimum degree  $\delta(G) \geq k$ , then  $px_k(G) \leq px_k(G[D]) + 1$ .*

*Proof.* Since  $D$  is a connected  $k$ -dominating set, every vertex  $v$  in  $\overline{D}$  has at least  $k$  neighbors in  $D$ . Let  $x = px_k(G[D])$ . We first color the edges in  $G[D]$  with  $x$  different colors from  $\{2, 3, \dots, x+1\}$  such that for every  $k$  vertices in  $D$ , there exists a proper tree in  $G[D]$  connecting them. Then we color the remaining edges with color 1.

Next, we will show that this coloring makes  $G$   $k$ -proper connected. Let  $S = \{v_1, v_2, \dots, v_k\}$  be any set of  $k$  vertices in  $G$ . Without loss of generality, we assume that  $\{v_1, \dots, v_p\} \subseteq D$  and  $\{v_{p+1}, \dots, v_k\} \subseteq \overline{D}$  for some  $p$  ( $0 \leq p \leq k$ ). For each  $v_i \in \overline{D}$  ( $p+1 \leq i \leq k$ ), let  $u_i$  be the neighbour of  $v_i$  in  $D$  such that  $\{u_{p+1}, \dots, u_k\}$  is a  $(k-p)$ -set. It is possible since  $D$  is a  $k$ -dominating set. Then the edges  $\{u_{p+1}v_{p+1}, \dots, u_kv_k\}$  together with the proper tree connecting the vertices  $\{v_1, \dots, v_p, u_{p+1}, \dots, u_k\}$  in  $G[D]$  induces a proper  $S$ -tree. Thus, we have  $px_k(G) \leq px_k(G[D]) + 1$ .  $\square$

Based on this theorem, we can give a lower bound and an upper bound on the  $k$ -proper index of a complete bipartite graph, whose proof is similar to Theorem 2.

**Theorem 10.** *For a complete bipartite graph  $K_{s,t}$  with  $t \geq s \geq k$ , we have  $2 \leq px_k(K_{s,t}) \leq 3$ .*

Let  $G$  be a complete bipartite graph. Using the techniques in Theorem 5, we can obtain the sufficient condition such that  $px_k(G) = 2$ .

**Theorem 11.** *For a complete bipartite graph  $K_{s,t}$  with  $t \geq s \geq 2(k-1)$ , we have  $px_k(K_{s,t}) = 2$ .*

*Proof.* We demonstrate a  $k$ -proper coloring of  $K_{s,t}$  with two colors 0 and 1, the same as Theorem 5. For completeness, we restate the coloring. Let  $U, W$  be the two partite sets of  $K_{s,t}$ , where  $U = \{u_1, u_2, \dots, u_s\}$  and  $W = \{w_1, w_2, \dots, w_t\}$ ,  $t \geq s \geq 2(k-1)$ . Denote a cycle  $C_s = u_1w_1u_2w_2 \dots u_sw_su_1$ . Let  $c(u_iw_i) = 0$  ( $1 \leq i \leq s$ ) and  $c(u_iw_j) = 1$  ( $1 \leq i \neq j \leq s$ ). And assign  $c(w_ru_i) = i \pmod{2}$  ( $1 \leq i \leq s, s < r \leq t$ ). Now, we show that for any  $k$ -subset  $S \subseteq V(K_{s,t})$ , there is a proper path  $P_S$  connecting all the vertices in  $S$ . Set  $W_1 = \{w_1, w_2, \dots, w_s\}$  and  $W_2 = \{w_{s+1}, \dots, w_t\}$  (if  $t > s$ ). Then  $S$  can be divided into three parts, i.e.,  $S = S_1 \cup S_2 \cup S_3$ , where  $S_1 = S \cap W_1$ ,  $S_2 = S \cap W_2$  and  $S_3 = S \cap U$ . Suppose  $|S_1| = p$ ,  $|S_2| = q$ , then  $p + q \leq k$ . If  $q = 0$ , the path  $P = u_1w_1u_2w_2 \dots u_sw_s$  is the proper path connecting  $S$ . If  $q \geq 1$ , set  $S_2 = \{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_q}\}$ , where  $s < \alpha_1, \alpha_2, \dots, \alpha_q \leq t$ . Let  $P = w_{\alpha_q}u_1w_1u_2w_2 \dots u_sw_s$ . Then consider the vertex set  $W'_S = \{w_{2i} : w_{2i} \in W_1 \setminus S_1\}$ . We have  $|W'_S| \geq s/2 - p \geq k - p - 1 \geq q - 1$ . So set  $|W'_S| = \ell$  and  $W'_S = \{w_{\beta_1}, w_{\beta_2}, \dots, w_{\beta_{q-1}}, \dots, w_{\beta_\ell}\}$ , where

$2 \leq \beta_1, \beta_2, \dots, \beta_\ell \leq s$  are even. Then we construct a path  $P_S$  by replacing the subpath  $u_{\beta_j} w_{\beta_j} u_{\beta_j+1}$  of  $P$  with  $u_{\beta_j} w_{\alpha_j} u_{\beta_j+1}$  (and  $u_s w_s$  with  $u_s w_{\alpha_j}$  if  $\beta_j = s$ ) for  $1 \leq j \leq q-1$ . Hence, the new path  $P_S$  is a proper path contains all the vertices of  $U$  so that  $P_S$  connects  $S_3$ . By the replacement we know that  $P_S$  also connects  $S_1$  as well as  $S_2$ . Thus we complete the proof.  $\square$

With the aids of Theorems 11 and 7, we can easily get the following, whose proof is similar to Theorem 8.

**Theorem 12.** *Let  $G = K_{n_1, n_2, \dots, n_r}$  be a complete multipartite graph, where  $r \geq 3$  and  $n_1 \leq n_2 \leq \dots \leq n_r$ . Set  $s = \sum_{i=1}^{r-1} n_i$  and  $t = n_r$ . If  $t \geq s \geq 2(k-1)$  or  $t \leq s$ , then we have  $px_k(G) = 2$ .*

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